What role does mathematics, particularly geometry and calculus, play in the projection of spherical surfaces onto two-dimensional planes? What is the application of this in the real world?

Mathematics Extended Essay

Abstract

The question on which my extended essay is focused is: **What role does mathematics, particularly geometry and calculus, play in the projection of spherical surfaces onto two-dimensional planes? What is the application of this in the real world?**

In this essay I first define the appropriate properties and terminology of the sphere to set a solid foundation for the more complex mathematics in later sections. I also introduce relevant theorems along with an introduction into the various types of projection. To discuss the mathematics of the projections themselves, I describe the projection process of each projection, then walk through the geometry and calculus involved to translate spherical coordinates on the three-dimensional plane into Cartesian coordinates on the two-dimensional plane, effectively demonstrating the role that mathematics plays in projection. For each projection, I also examine its properties and use my resulting equations to exemplify a potential projection of a spherical point. To ensure that I have a variety of examples, I discuss projections that preserve different characteristics of the sphere's surface, as well as both direct projection onto a two-dimensional plane and projection first onto a three-dimensional surface before flattening.

I cultivate a higher understanding of the projection concept by underscoring and justifying the various applications of them in the real world. I summarize these applications in my conclusion, while also discussing the possible limitations of the mathematics in this paper when it comes to applying it.

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1. Introduction

The significance of cartography to humankind can be exhibited by its history of more than 5,000 years. Aside from their prominence in the advancement of navigation and communication, maps have long been an outlet for human understanding of the fascinating world around us on an otherwise intangible scale.

Even in the modern era, maps of all scales and sizes are often plastered across classroom walls and used almost on a daily basis for basic navigation. Relevant in a scientific, historic, and even artistic perspective, cartography is a powerful tool prevalent in the lives of millions. Given this dually immense historical and modern-day significance of cartography, it wasn't long before I started to question the connection between our three-dimensional, spheroid planet Earth and the flat, two-dimensional planes that maps appear in. Surprisingly, the applications of calculus just above my current level are the focus of much of the map projection process.

In this essay, I shall investigate the use of mathematics, particularly geometry and calculus, in the projection of spherical surfaces onto two-dimensional surfaces. Relatedly, I will explore the application of this projection process in the context of navigation, cartography and map projection of the Earth.

2. The Sphere

2.1. Properties and Terminology

In order to assess the mathematics behind the projection of three-dimensional spherical surfaces onto two-dimensional planes, it is first necessary to be acquainted with the sphere and its properties. Throughout this paper, the *unit sphere*, otherwise known as a locus of points of a constant distance of one unit from a given origin on a three-dimensional plane, will be used as a model for projection, along with terminology pertaining to the various properties of spheres. Thus, we now introduce the relevant terminology.

A *great circle* is the intersection of a two-dimensional plane that passes through the center of a sphere, O, and the sphere itself. A fixed great circle referenced as the equator (E) splits a sphere into two hemispheres: a northern hemisphere and a southern hemisphere. Corresponding to E are two *poles* N and S, the two points in the northern and southern hemispheres, respectively, that are furthest from E.

Given a unit sphere with origin O mapped on a three-dimensional Cartesian coordinate system, we establish a point P on the surface of the sphere (P being a point other than the sphere's poles) specified by Cartesian coordinates $(x,\ y,\ z)$. The *meridian*, M_P, of P is half of the great circle containing P and terminated by the poles. We also establish a reference point R on E, along with its meridian M_R. The scenario thus far is depicted below in Diagram 1:

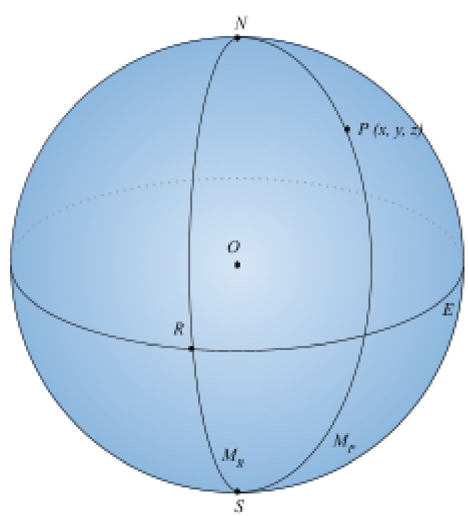

Diagram 1: Setup of the Unit Sphere (O, E, N, S, P, M_P, R, M_R)

Point P, however, can also be specified by latitude and longitude in reference R.

Latitude (θ) is the angle measure from the intersection of M_P and E, to P. This measure can be either positive or negative, depending on whether P falls above or below E. For the purposes of this paper, we will limit our scope to a positive θ. The locus of points with a constant latitude is a *parallel of latitude*. *Longitude* (ϕ) is the angle measure along the equator from R to the intersection of M_P and E. As with θ, ϕ can also be either positive or negative, depending on whether it is measure in the clockwise or counterclockwise direction, respectively, along E from R as viewed from S. Within this paper, our scope will be limited to a positive ϕ—a clockwise rotation as viewed from S. Latitude and longitude in context of the above description, along with the parallel of latitude at the latitude of P, are illustrated below in Diagram 2:

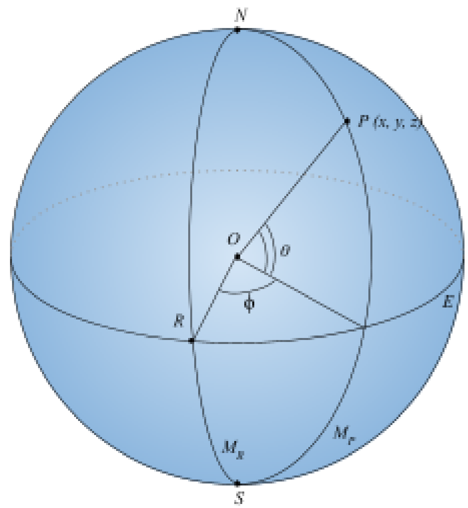

Diagram 2: Latitude (θ) and Longitude (ϕ) of P

2.2. Parametrization of the Sphere

As discussed previously, P can be specified by Cartesian coordinates $(x,\ y,\ z)$, but also by angle measure with latitude and longitude (θ, ϕ). Inherently, projection of a three-dimensional spherical surface onto a two-dimensional plane involves the relaying of three-dimensional Cartesian coordinates into two-dimensional Cartesian coordinates. This means that $(x,\ y,\ z)$ needs to have a common translation to $(u,\ v)$—the specifying coordinates of the two-dimensional plane. This is done by defining all variables as a function of θ and ϕ; thus, we now parametrize the Cartesian variables of the unit sphere as functions of θ and ϕ.

Taking our point P at a positive $(x,\ y,\ z)$, we first draw a line straight down (cutting through the sphere) to the flat xy-plane (to the point $P_2(x,\ y,\ 0)$), creating a right triangle. This is illustrated in Diagram 3, while the triangle is highlighted in Diagram 4:

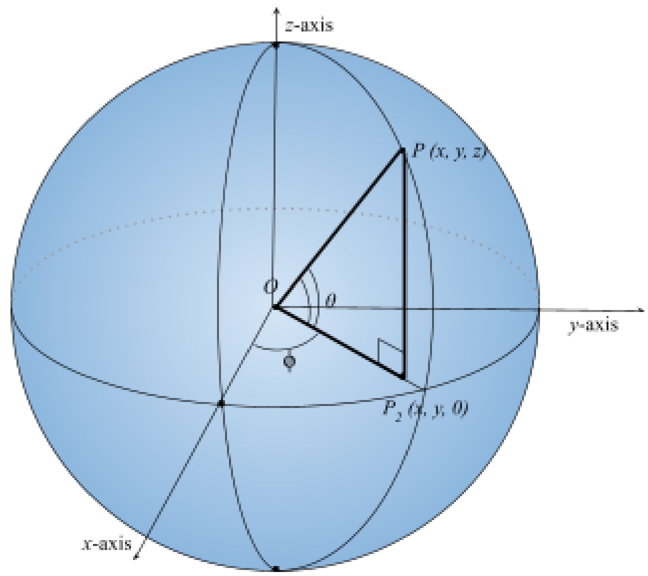

Diagram 3: Triangle (focus on θ) in the Sphere

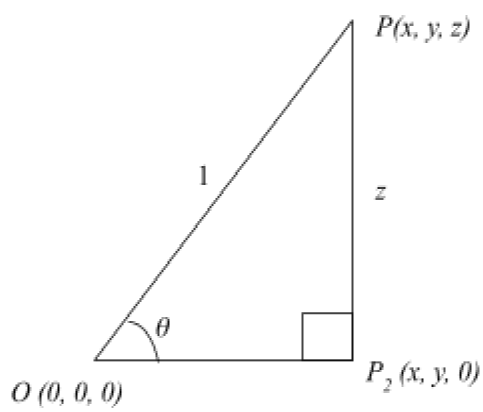

Diagram 4: Triangle (focus on θ)

As highlighted in Diagram 4, the hypotenuse of the created triangle is 1, because it is a radius of the unit sphere. Additionally, the leg opposite of ∠θ has length z because, as seen in Diagram 3, it corresponds with the height on the z-axis. By trigonometric ratios, then simplification:

$$\sin \theta = \frac{z}{1}$$

$$z = \sin \theta$$

The length of the leg adjacent to ∠θ can also be found in this way:

$$\cos\theta = \frac{\overline{OP_2}}{1}$$
$$\overline{OP_2} = \cos\theta$$

To attain the parametrization of x and y, we create another right triangle, this time focusing on ϕ rather than θ. This new triangle lies flat on the xy-plane, created by drawing a line from the point $P_2(x, y, 0)$ to $P_1(x, 0, 0)$, and is shown in Diagrams 5 and 6:

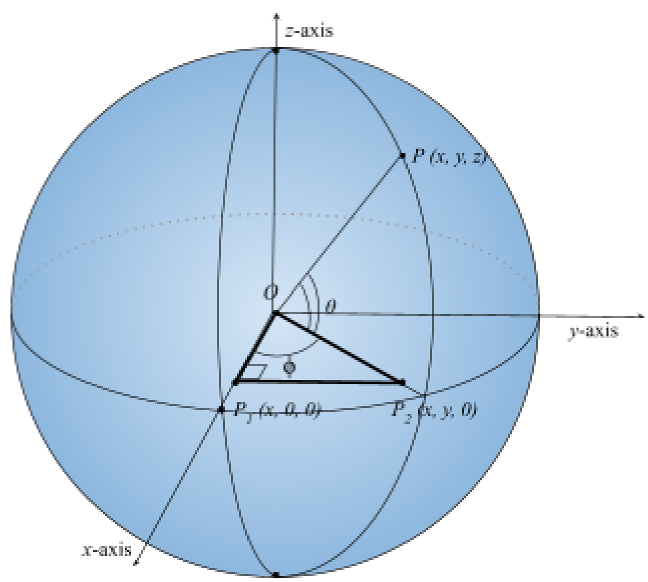

Diagram 5: Triangle (focus on ϕ) in the Sphere

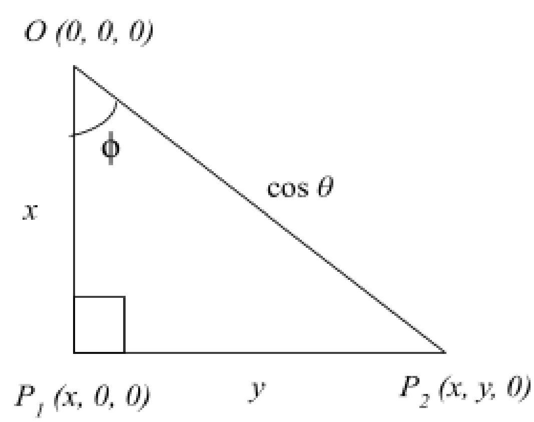

Diagram 6: Triangle (focus on ϕ)

The triangle leg adjacent to $\angle\phi$ has length x because it corresponds with the x-axis, while the leg opposite of $\angle\phi$ has length y because it corresponds with the y-axis, both of which are depicted in Diagram 5. Additionally, the hypotenuse extending from O to P_2 is shared with the leg adjacent to $\angle\theta$ on the first triangle we created. We found earlier that this had length $cos\theta$.

Using these values, we find using trigonometric ratios:

$$cos\,\phi\ = \frac{x}{cos\theta}$$
$$sin\,\phi\ = \frac{y}{cos\theta}$$

Simplifying, we attain the parametrized x and y:

$$x\ =\ cos\,\phi\, cos\,\theta$$
$$y\ =\ sin\,\phi\, cos\,\theta$$

Thus, we now attain the parametrization of the Cartesian coordinates of the unit sphere. Although we will be working with spherical coordinates $(\phi,\ \theta)$ in this paper, knowing how to translate between these coordinates and Cartesian $(x,\ y,\ z)$ is instrumental in understanding the full projection from three dimensions to two dimensions.

3. An Introduction to Projection

3.1. Theorema Egregium

Before discussing the mathematics of three-dimensional to two-dimensional projection itself, it is necessary to understand the distortion factor: it is impossible to display a three-dimensional surface in two dimensions without distortion. While it is possible to retain certain properties, such as area, distance, or angle, not all properties can be preserved simultaneously. This was proved by Carl Friedrich Gauss with his *Theorema Egregium* in 1828; although his proof is outside the scope of this paper, it can be described with a more mundane analogy. Taking an orange peel (a three-dimensional spherical surface), it is impossible to flatten it (into a two-dimensional plane) without tearing or stretching it. Just like with the orange peel, for an untorn projection of any sphere, it is inevitable that its surface be "stretched", causing distortion of certain properties.

3.2. Types of Projection

Due to the impossible nature of a perfect projection, one defining factor of a given projection centers around which properties are retained. For example, as an application in the real world, certain maps of the Earth retain the relative areas of land masses, while others retain relative distances between points on the Earth. Other maps that retain angle are quintessential for navigation with a compass. Throughout this paper, the specific details of projections as they pertain to the application of cartography will be discussed.

Another differentiating factor between projections is the shape of the two-dimensional surface onto which the sphere is projected, also known as a *developable surface*. Formally, developable surfaces are known as three-dimensional surfaces that can be flattened or "unrolled" into a flat surface without the stretching or tearing that would characterize an attempt to flatten a sphere directly. Projections work by first projecting the surface of the sphere onto a three-dimensional developable surface before "unrolling" the new projection, leaving a distorted version of the original surface on a two-dimensional plane. Developable surfaces include the cylindrical and conical surfaces, as well as direct projections onto two-dimensional planar surfaces.

Between these two factors, there are a myriad of possible combinations; a cylindrical distance-preserving map, a cylindrical angle-preserving map, or a conical angle-preserving map are some examples of projections. In this paper, we will be exploring the planar gnomonic projection (projects directly onto a plane) and three types of conformal projections (first projects onto another three-dimensional surface).

4. Planar Gnomonic Projection

Let us start with the *planar gnomonic projection*, a direct projection onto a two-dimensional plane. This projection is a *central projection*—a projection from the center of the sphere—and shows less than a hemisphere. The setup of this projection involves a flat plane tangent to the sphere (for this paper we take the plane tangent to the north pole N and name it T). The projection works by drawing the line between the center of the sphere and a given point on the surface of the sphere, and finding the intersection of the line with T. The intersection point serves as the two-dimensional projection of the originally three-dimensional point. Diagram 7 provides a visual representation of a projection of three-dimensional points Q_3 and P_3 on the spherical surface onto two-dimensional Q_2 and P_2, respectively, on the flat plane.

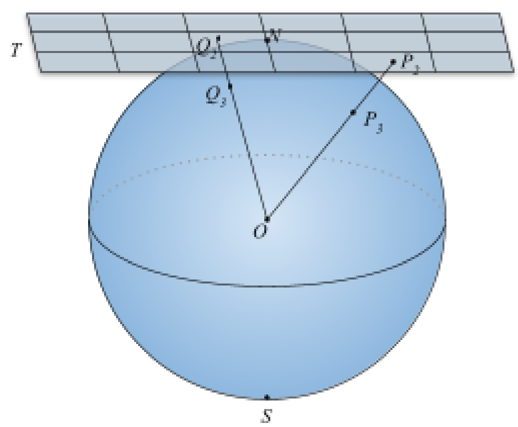

Diagram 7: Planar Gnomonic Projection of Q_3 and P_3

The most notable property of the planar gnomonic projection is that it maps great circles onto straight lines on T. This is because, as defined in Section 2.1, a great circle is by definition the intersection of a two-dimensional plane (let's define the plane as G) that passes through the center of a sphere and the sphere itself. This means that, in conducting the gnomonic projection, any line drawn between the center of the sphere and a point on the great circle (on the surface of the sphere) is contained in G. Since the lines drawn during the projection process for any great circle all lie in G, their projection on the T is the intersection of G and T. The intersection of any two non-parallel planes is a straight line. For clarification, this concept is illustrated in Diagram 8, where the final projection of a straight line is highlighted in yellow:

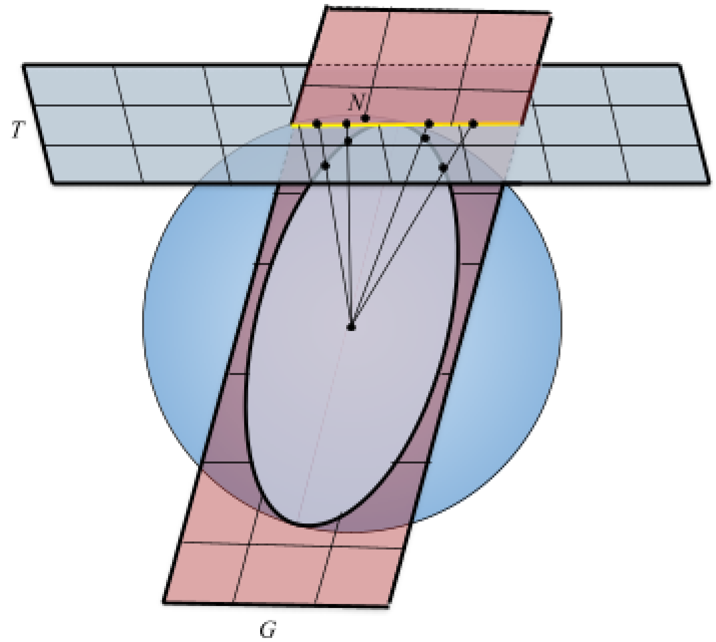

Diagram 8: Great Circle Mapped to Straight Line

This has significant implications as an application. On a sphere's surface, a great circle marks the shortest distance between any given two points. Consequently, for navigators on the Earth's surface, the shortest distance between two points is marked by drawing the straight line between them on a map.

Great circles marking the shortest distance between points on a sphere can be proven with geometry and calculus:

> Given two points P_A and P_B on a sphere (of radius R), the straight line between them is $\overline{P_AP_B}$. Then, let us define a plane I intersecting the sphere that includes P_A and P_B. This scenario, as well as the circular cross-section (of radius r and center o) of the sphere and I are depicted in Diagram 9. The arc on the edge of the circular cross-section between P_A and P_B is P_AP_B.

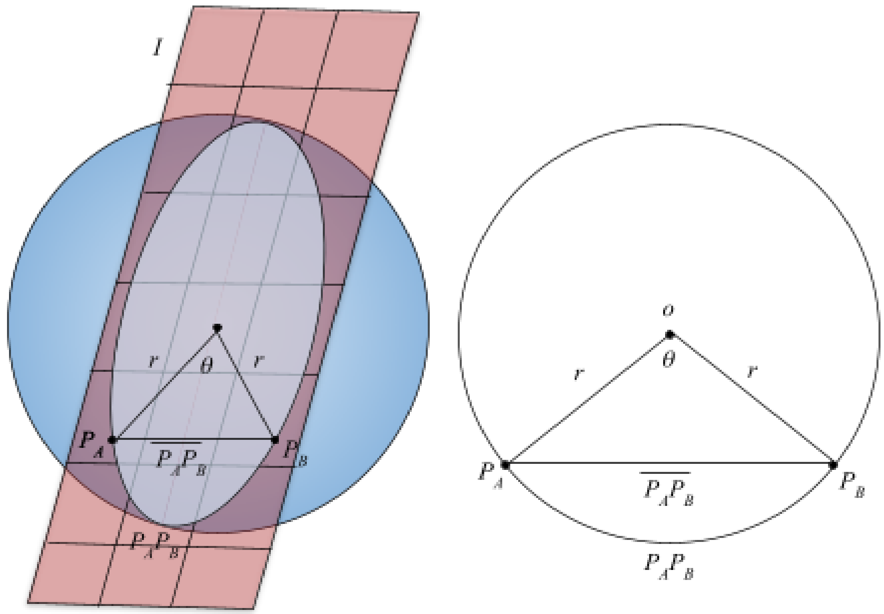

Diagram 9: P_A and P_B Plane Intersection and Cross-Section

We now aim to find the value of r which minimizes P_AP_B; we minimize P_AP_B because, in the application of the Earth, we can only travel on its surface.

Firstly, we establish that $\frac{\overline{P_AP_B}}{2} \leq r \leq R$. $r = R$ if I passes through the center of the sphere, because that would make the cross-section a great circle with its radius equal to the radius of the sphere. However, it can also be possible to have $r < R$ if I does not intersect the center of the sphere, and the circular cross-section has a smaller radius than the sphere. Additionally, the maximum distance (or value of $\overline{P_AP_B}$) between P_A and P_B is the diameter ($2r$) of the circular cross-section, meaning $2r \geq \overline{P_AP_B}$. Simplifying, we get $\frac{\overline{P_AP_B}}{2} \leq r$.

Focusing on the circular cross-section, we use the arc length formula $s(r) = r\theta$, where $s(r)$ is the arc P_AP_B, r is the radius, and θ is the angle measure between P_A and P_B.

The law of cosines allows us to claim that, for triangle ΔP_AP_Bo:

$$(\overline{P_AP_B})^2 = 2r^2 - 2r^2 cos\theta$$

Solving for θ, we get

$$2r^2 cos\theta = 2r^2 - (\overline{P_A P_B})^2$$

$$cos\theta = \frac{2r^2-(\overline{P_A P_B})^2}{2r^2}$$

$$\theta = cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})$$

We can now plug this value of θ into the arc length formula:

$$s(r) = r \times cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})$$

To minimize $P_A P_B$, we find the first and second derivatives of $s(r)$.

Because $s(r)$ consists of s, r, and $\overline{P_A P_B}$ as variables, it is necessary to take a partial derivative with respect to r, as that is the variable we are looking to minimize. As a result, $\overline{P_A P_B}$ will be regarded as a fixed number.

First we apply the product rule.

$$s'(r) = \frac{\partial}{\partial r}(r)(cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})) + \frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2}))r$$

$\frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2}))$ requires further calculation.

We briefly substitute $u = \frac{2r^2-(\overline{P_A P_B})^2}{2r^2}$.

Knowing that $\frac{d}{du}cos^{-1}(u) =- \frac{1}{\sqrt{1-u^2}}du$, we substitute u back in:

$$\frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})) =- \frac{1}{\sqrt{1-(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})^2}} \times \frac{\partial}{\partial r}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})$$

Both factors here need to be simplified further.

Firstly, with $- \frac{1}{\sqrt{1-(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})^2}}$, we square the fraction under the square root and transform the radicand into a single fraction by matching the denominator of the 1, allowing the overall fraction to be simplified.

$$- \frac{1}{\sqrt{1-\frac{(4r^4-4r^2(\overline{P_A P_B})^2+(\overline{P_A P_B})^4)}{4r^4}}}$$

$$-\frac{1}{\sqrt{\frac{4r^4}{4r^4}-\frac{(4r^4-4r^2(\overline{P_AP_B})^2+(\overline{P_AP_B})^4)}{4r^4}}}$$

$$-\frac{1}{\sqrt{\frac{4r^4-(4r^4-4r^2(\overline{P_AP_B})^2+(\overline{P_AP_B})^4)}{4r^4}}}$$

$$-\frac{1}{\frac{\sqrt{4r^2(\overline{P_AP_B})^2-(\overline{P_AP_B})^4}}{\sqrt{4r^4}}}$$

Our final result is: $-\frac{2r^2}{\overline{P_AP_B}\sqrt{4r^2-(\overline{P_AP_B})^2}}$.

Now, we simplify the other factor, $\frac{\partial}{\partial r}(\frac{2r^2-(\overline{P_AP_B})^2}{2r^2})$. We first apply the quotient rule, then simplify the resulting fraction.

$$\frac{2r^2(\frac{\partial}{\partial r}(2r^2-(\overline{P_AP_B})^2))-(2r^2-(\overline{P_AP_B})^2)(\frac{\partial}{\partial r}(2r^2))}{(2r^2)^2}$$

$\frac{\partial}{\partial r}(2r^2) = 4r$, applying the power rule.

$\frac{\partial}{\partial r}(2r^2 - (\overline{P_AP_B})^2) = 4r$, for the same reason, and $-(\overline{P_AP_B})^2$ can be disregarded with $\overline{P_AP_B}$ being considered a fixed value.

$$\frac{2r^2(4r)-(2r^2-(\overline{P_AP_B})^2)(4r)}{4r^4}$$

$$\frac{8r^3-(8r^3-4r(\overline{P_AP_B})^2)}{4r^4}$$

$$\frac{4r(\overline{P_AP_B})^2}{4r^4}$$

Our final result is: $\frac{(\overline{P_AP_B})^2}{r^3}$.

Now that both factors have been simplified, we attain:

$$\frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_AP_B})^2}{2r^2})) =- \frac{2r^2}{\overline{P_AP_B}\sqrt{4r^2-(\overline{P_AP_B})^2}} \times \frac{(\overline{P_AP_B})^2}{r^3}$$

This can be simplified to:

$$\frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_AP_B})^2}{2r^2})) =- \frac{2\overline{P_AP_B}}{r\sqrt{4r^2-(\overline{P_AP_B})^2}}.$$

Finally, we state that $\frac{\partial}{\partial r}(r) = 1$.

We can now substitute $\frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2}))$ and $\frac{\partial}{\partial r}(r)$ into our original product rule:

$$s'(r) = 1 * (cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})) + (-\frac{2\overline{P_A P_B}}{r\sqrt{4r^2-(\overline{P_A P_B})^2}}) * r$$

The first derivative of $s(r)$ is:

$$s'(r) = cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2}) - \frac{2(\overline{P_A P_B})}{\sqrt{4r^2-(\overline{P_A P_B})^2}}$$

Now we find the second derivative of $s(r)$ by taking the derivative of $s'(r)$. Using the sum and difference rule, we know that:

$$s''(r) = \frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})) - \frac{\partial}{\partial r}(\frac{2(\overline{P_A P_B})}{\sqrt{4r^2-(\overline{P_A P_B})^2}})$$

From our calculation of $s'(r)$, we already know that

$$\frac{\partial}{\partial r}(cos^{-1}(\frac{2r^2-(\overline{P_A P_B})^2}{2r^2})) =- \frac{2(\overline{P_A P_B})}{r\sqrt{4r^2-(\overline{P_A P_B})^2}}$$

$$\frac{\partial}{\partial r}(\frac{2(\overline{P_A P_B})}{\sqrt{4r^2-(\overline{P_A P_B})^2}})$$

From here, we plug back into our final equation for $s''(r)$.

$$s''(r) =- \frac{2(\overline{P_A P_B})}{r\sqrt{4r^2-(\overline{P_A P_B})^2}} + \frac{8cr}{(4r^2-(\overline{P_A P_B})^2)^{\frac{3}{2}}}$$

Lastly, we find a common denominator and simplify.

$$-\frac{2(\overline{P_A P_B})(4r^2-(\overline{P_A P_B})^2)}{r(4r^2-(\overline{P_A P_B})^2)^{\frac{3}{2}}} + \frac{8(\overline{P_A P_B})r^2}{r(4r^2-(\overline{P_A P_B})^2)^{\frac{3}{2}}}$$

$$\frac{-8(\overline{P_A P_B})r^2+2(\overline{P_A P_B})^3+8(\overline{P_A P_B})r^2}{r(4r^2-(\overline{P_A P_B})^2)^{\frac{3}{2}}}$$

The second derivative of $s(r)$ is:

$$s''(r) = \frac{2(\overline{P_A P_B})^3}{r(4r^2-(\overline{P_A P_B})^2)^{\frac{3}{2}}}$$

Now that we have $s'(r)$ and $s''(r)$, we can tell that

I. $s(r)$ is continuous on $[\frac{\overline{P_A P_B}}{2}, \infty)$

II. $s(r)$ is twice-differentiable on $(\frac{\overline{P_A P_B}}{2}, \infty)$

This allows us to claim that, when $r > \frac{\overline{P_A P_B}}{2}$

I. $4r^2 - (\overline{P_A P_B})^2 > 0$, because $4(\frac{\overline{P_A P_B}}{2})^2 - (\overline{P_A P_B})^2 = 0$

II. $s''(r) > 0$, because $s''(r)$ is always positive when $4r^2 - (\overline{P_A P_B})^2 > 0$

III. $s'(r)$ increases, because $s''(r) > 0$ signifies a positive slope for $s'(r)$

Although we know that $s'(r)$ increases, in order to assess the minimum value of $s(r)$ we need to know if $s'(r)$ is positive or negative when $r > \frac{\overline{P_A P_B}}{2}$ by taking the limit of $s'(r)$ as r approaches each boundary.

$$\lim_{r \to \frac{\overline{P_A P_B}}{2}^+} s'(r) = cos^{-1}\left(\frac{2(\frac{\overline{P_A P_B}}{2})^2 - (\overline{P_A P_B})^2}{2(\frac{\overline{P_A P_B}}{2})^2}\right) - \frac{2\overline{P_A P_B}}{\sqrt{4(\frac{\overline{P_A P_B}}{2})^2 - (\overline{P_A P_B})^2}}$$

$$= cos^{-1}\left(\frac{\frac{\overline{P_A P_B}^2}{2} - (\overline{P_A P_B})^2}{\frac{\overline{P_A P_B}^2}{2}}\right) - \frac{2\overline{P_A P_B}}{\sqrt{(\overline{P_A P_B})^2 - (\overline{P_A P_B})^2}}$$

$$= cos^{-1}(-1) - \frac{2\overline{P_A P_B}}{\sqrt{0}}$$

$$= \pi - \infty$$

$$= -\infty$$

$$\lim_{r \to \infty} s'(r) = cos^{-1}\left(\frac{2(\infty)^2 - (\overline{P_A P_B})^2}{2(\infty)^2}\right) - \frac{2\overline{P_A P_B}}{\sqrt{4(\infty)^2 - (\overline{P_A P_B})^2}}$$

$$= cos^{-1}(1) - \frac{2\overline{P_A P_B}}{\sqrt{\infty}}$$

$$= 0 - 0$$

$$= 0$$

Knowing that $s'(r)$ increases when $r > \frac{\overline{P_A P_B}}{2}$, $\lim_{r \to \frac{\overline{P_A P_B}}{2}^+} s'(r) = -\infty$, and $\lim_{r \to \infty} s'(r) = 0$, we conclude that $s'(r) < 0$ when $r > \frac{\overline{P_A P_B}}{2}$. This tells us that $s(r)$ decreases between when $r > \frac{\overline{P_A P_B}}{2}$. $s(r)$ is minimized when r is maximized, and because our limits for r are $\frac{\overline{P_A P_B}}{2} \leq r \leq R$, $s(r)$ is at its minimum when $r = R$—this is the great circle case.

Thus, we prove that great circles are the shortest route between any two points on the surface of a sphere, and demonstrate the applicability of the gnomonic projection by highlighting the correlation between great circles being the shortest route, and the mapping of great circles into straight lines with the gnomonic projection.

5. Conformal Map Projection

The next family of projections is a *conformal map projection*. This type of projection preserves angles between any two curves on the surface of a sphere. This relationship is highlighted below in Diagram 10—regardless of the difference between $\Delta\Phi$ and Δu or $\Delta\theta$ and Δv, the conformal projection preserves $\angle\alpha$ of the intersection between any two curves. We will be highlighting a conformal map projection onto three different developable surfaces: the plane, the cylinder, and the cone.

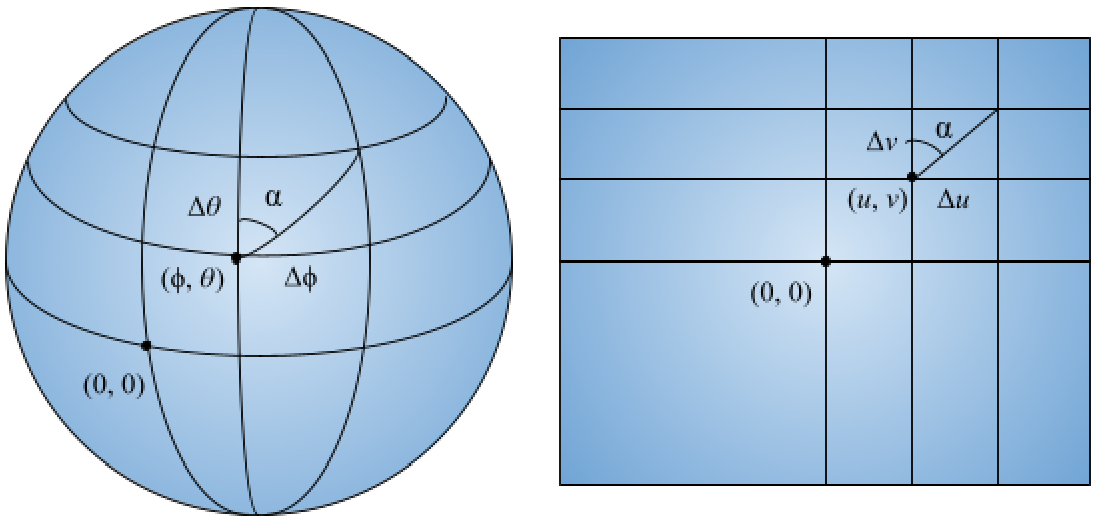

Diagram 10: Conformal Projection, Preservation of $\angle\alpha$

5.1. Stereographic Projection

We will start with the stereographic projection because, out of the three conformal projections we will explore, it is most similar to the gnomonic projection we have explored thus far. The stereographic projection also projects from the sphere directly onto a tangent plane, but it is not a central projection; the projection is instead from the opposite pole. The key difference between planar gnomonic and stereographic projections is that stereographic projections, as conformal projections, preserve the angle between intersecting curves. The projection we will examine (a plane T tangent to the north pole N and projecting from the south pole S of a unit sphere) is depicted in Diagram 11, displaying projection of P_3 to P_2:

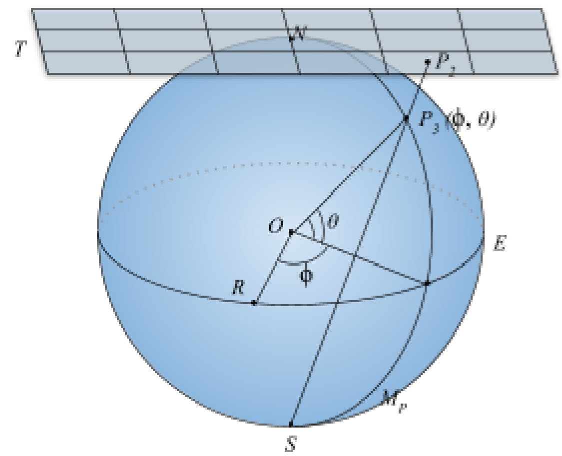

Diagram 11: Stereographic Projection

As defined in Section 2.1, point P can be specified by latitude θ and longitude ϕ in reference to a point R on the equator E. Our aim in the mathematics of this projection is to find a way to attain the projected Cartesian coordinates (u, v) on T of a point P_3 given in terms of θ and ϕ. For this paper, $\theta \in (0, \frac{\pi}{2})$ and $\phi \in (0, \frac{\pi}{2})$.

We will first focus on θ, and visualize the projection from the side of the globe so that the sphere appears as a circle and T appears as a line (Diagram 12).

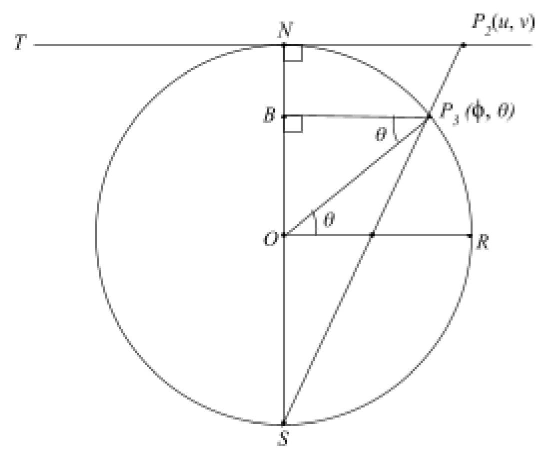

Diagram 12: Stereographic Projection Side View

By Alternate Interior Angles Theorem, we know that $\angle P_3OR \cong \angle BP_3O$, given that B is defined such that $\overline{BP} \parallel \overline{OR}$. We also know that $\triangle SBP_3$ and $\triangle SNP_2$ are similar by Angle-Angle Similarity Theorem, as they share in common $\angle BSP_3$ and

$\angle NSP_2$ and both have right angles. This allows us to solve for the length of $\overline{NP_2}$, which allows us to attain the final Cartesian coordinates of P_2.

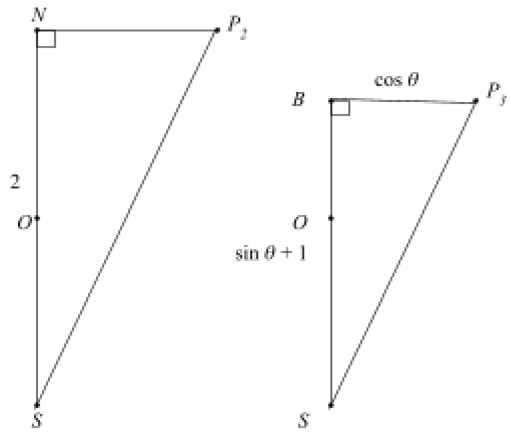

Diagram 13: Similar Triangles ΔSBP_3 and ΔSNP_2

Because $\angle P_3OR = \theta$ and $\angle P_3OR \cong \angle BP_3O$, we know that $\angle BP_3O = \theta$. Thus, we use trigonometric ratios to find $\overline{BP_3}$:

$$cos\,\theta = \frac{\overline{BP_3}}{\overline{OP_3}}$$

Since $\overline{OP_3}$ is equal to the radius of the unit sphere (1),

$$\overline{BP_3} = cos\,\theta$$

Similarly, we can conclude that

$$sin\,\theta = \frac{\overline{BO}}{\overline{OP_3}}$$

$$\overline{BO} = sin\,\theta$$

Since $\overline{OS}$ is a radius of the sphere,

$$\overline{OS} = 1$$

Therefore,

$$\overline{BS} = \overline{BO} + \overline{OS} = sin\,\theta + 1$$

Additionally, since $\overline{NS}$ is a diameter of the sphere,

$$\overline{NS} = 2$$

The ratios of the corresponding sides of ΔSBP_3 and ΔSNP_2 are equal, so we can solve for $\overline{NP_3}$:

$$\frac{\overline{NP_3}}{2} = \frac{cos\theta}{sin\theta + 1}$$

$$\overline{NP}_3 = \frac{2cos\theta}{sin\theta+1} = \frac{2}{tan\theta+sec\theta}$$

Now that we have a value for $\overline{NP}_3$, we need to take into account ϕ. To do this, we take a view of T from the top. The point of tangency (N) is $(0, 0)$ in Cartesian coordinates, and the meridian of R maps onto the negative v-axis. As a result, a point P_3 with $\phi \in (0, \frac{\pi}{2})$ rotating clockwise from R as viewed from S would project to a point P_2 in the bottom right quadrant of T (Diagram 14).

From this, we can gather that:

$$sin\,\phi = \frac{u}{\overline{NP}_2}$$

$$cos\,\phi = \frac{-v}{\overline{NP}_2}$$

And thus we attain Cartesian coordinates on T, u and v, as functions of spherical coordinates ϕ and θ:

$$u = sin\,\phi * \overline{NP}_2 = sin\phi * \frac{2}{tan\theta+sec\theta}$$

$$v =- cos\,\phi * \overline{NP}_2 =- cos\,\phi * \frac{2}{tan\theta+sec\theta}$$

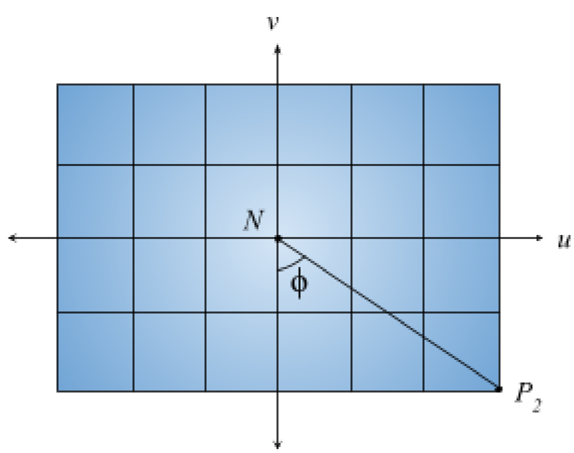

Diagram 14: Plane T Top View

This can be demonstrated with a potential spherical coordinate $(\phi, \theta) = (\frac{\pi}{3}, \frac{\pi}{4})$.

$$u = sin\,(\frac{\pi}{3}) * \frac{2}{tan(\frac{\pi}{4})+sec(\frac{\pi}{4})} = \frac{\sqrt{3}}{2} * \frac{2}{1+\frac{2}{\sqrt{2}}} = \frac{\sqrt{6}}{\sqrt{2}+2} \approx 0.717$$

$$v =- cos\,(\frac{\pi}{3}) * \frac{2}{tan(\frac{\pi}{4})+sec(\frac{\pi}{4})} =- \frac{1}{2} * \frac{2}{1+\frac{2}{\sqrt{2}}} =- \frac{\sqrt{2}}{\sqrt{2}+2} \approx- 0.414$$

Spherical coordinates $(\phi, \theta) = (\frac{\pi}{3}, \frac{\pi}{4})$ project to Cartesian coordinates $(0.717, \ - 0.414)$.

As it is conformal, the stereographic projection maps all circles on the sphere to circles on T. This creates useful applications in creating maps of not just the Earth, but especially of other planetary bodies. As an example, the surface of the moon is covered by circular craters. The stereographic projection is useful in preserving the circular shape of these craters on the three-dimensional spherical surface, on the two-dimensional flat plane.

5.2. Cylindrical Projection

Our second conformal projection is the cylindrical projection, better known as the famous (Standard) Mercator projection. The setup for this projection includes a hollow cylinder that is tangent to the equator E of a unit sphere. This projection first maps the sphere onto the inside of the cylinder before "cutting it" vertically and "rolling it out" onto a rectangular two-dimensional plane T. The projection is *not* from the center of the sphere, O. This is because the projection is not from a specific point. Instead, the sphere acts as a "balloon", and as it "inflates" and intersects with the cylinder, it "clings" to the inside of the cylinder. For the purposes of this paper, the vertical line we will "cut" to unroll the cylinder is the antipode of our reference point R (where $(\phi, \theta) = (0, 0)$) so that R itself ends up in the center of T. The projection process of P_3 (on the surface of the sphere) to P_2 (on the inside of the cylinder and unrolled on T), R_3 to R_2 (the reference point), and E_3 to E_2 (the equator) following the Mercator projection is shown in Diagram 15.

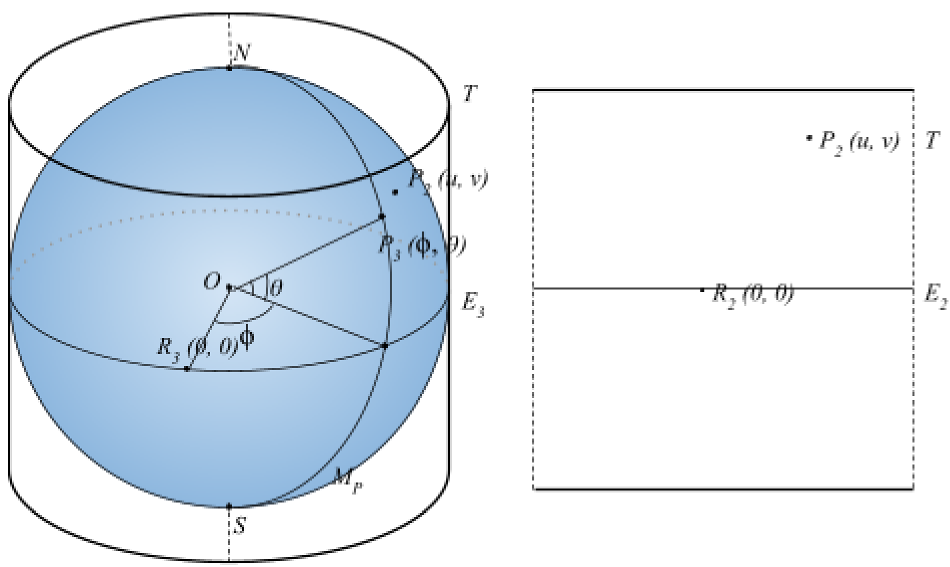

Diagram 15: Cylindrical Projection (Project and Unroll)

There are some properties of the cylindrical projection we must consider. Because the unit sphere has a radius of 1, E (a circumference of the sphere) has a radius 2π. The cylinder is tangent to E, so its width when unrolled into a rectangle is also 2π. Additionally, we assume that meridians map to vertical lines on T while parallels of latitude map to horizontal lines. This is because of preservation of angle—meridians have constant longitude while parallels have constant latitude, and the nature of a conformal projection preserves this feature.

As with the stereographic projection, our aim of the mathematics of this projection is to find a way to attain the projected Cartesian coordinates (u, v) on T of a point given in terms of θ and ϕ while preserving the angle $\angle\alpha$ of the intersection between any two curves. Rather than working with the projection of a single point, we will take the projection of a rectangular space, entailing the use of not only a point $P_3(\phi, \theta)$ but also a small angular change $\Delta\phi$ and $\Delta\theta$ from P_3. This projects onto a point $P_2(u, v)$ with small change in length Δu and Δv on T. This concept is highlighted in Diagram 10.

We start by looking at the radius of the circle that makes up the parallel of latitude of θ. It is important to differentiate that this is the parallel of θ and not $\theta + \Delta\theta$. As shown in Diagram 16, E is the equator, E_θ is the parallel of latitude at θ, O is the center of the sphere, O_θ is the center of E_θ, P_E is the point $(\phi, 0)$ (shares the latitude of P_3 but intersects with E), and R is the reference point from which ϕ and θ are measured.

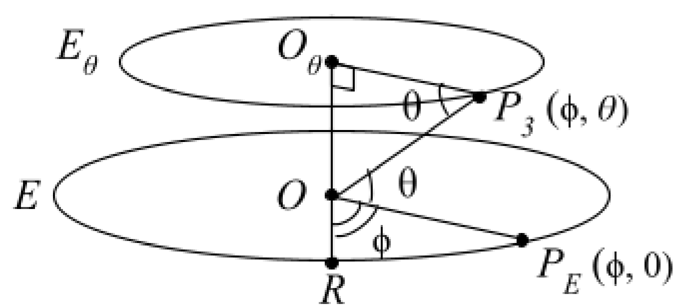

Diagram 16: Parallel of Latitude of θ

By Alternate Interior Angles Theorem, we know that $\angle O_\theta P_3 O \cong \angle P_3 O P_E$, since $E \parallel E_\theta$. Also, $\overline{OP_3} = 1$ as a radius of the unit sphere.

Considering that $\angle OO_\theta P_3$ is a right angle,

$$cos\,\theta = \frac{\overline{O_\theta P_3}}{\overline{OP_3}} = \frac{\overline{O_\theta P_3}}{1} = \overline{O_\theta P_3}$$

Now that we know this, we can focus on finding the lengths of the edges of the rectangular area created by $\Delta\phi$ and $\Delta\theta$. This is important because these lengths correspond to Δu and Δv on T. As clarified in Diagram 17, it's important to note that $\Delta\phi$ and $\Delta\theta$ in this case denote change in angle measure, not the length of the arc itself, which we will now find. Δu and Δv, however, denote length on T rather than angle measure. Diagram 17 introduces new points $P_{\Delta\phi}$, $P_{\Delta\theta}$, P_Δ that make up the vertices of the rectangular area (along with P_3).

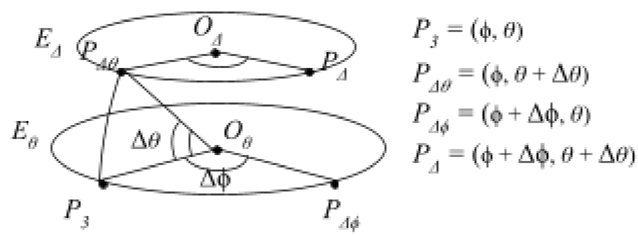

Diagram 17: Close-up of E_θ

Using the arc length formula $s = r\theta$, we can attain $\overline{P_3P_{\Delta\theta}}$ (or the edge of the rectangular area parallel to the meridians and corresponding to $\Delta\theta$) and $\overline{P_3P_{\Delta\phi}}$ (or the edge parallel to the parallels of latitude and corresponding to $\Delta\phi$).

$$\overline{P_3P_{\Delta\phi}} = (\overline{O_\theta P_3})(\Delta\phi) = \Delta\phi\, cos(\theta)$$
$$\overline{P_3P_{\Delta\theta}} = (\overline{OP_3})(\Delta\theta) = (1)(\Delta\theta) = \Delta\theta$$

Also, since $\overline{P_3P_{\Delta\theta}} \cong \overline{P_{\Delta\phi}P_\Delta}$,

$$\overline{P_{\Delta\phi}P_\Delta} = \Delta\theta$$

Using these results, we take into account $\angle\alpha$ and form an equation involving $\angle\alpha$ in terms of ϕ and θ. $\angle\alpha$ (found in Diagram 17 as $\angle P_{\Delta\theta}P_3P_\Delta$) needs to be preserved with the projection. Using the Alternate Interior Angles Theorem, we know that $\angle a$ is congruent to $\angle P_3P_\Delta P_{\Delta\phi}$. Additionally, although $\angle P_3P_{\Delta\phi}P_\Delta$ on the spherical surface is not a right angle, its projection on the two-dimensional surface is. Since the conformal projection aims to preserve angle, we can treat $\angle P_3 P_{\Delta\phi} P_{\Delta}$ as a right angle. As a result,

$$cot\ \angle P_3 P_{\Delta} P_{\Delta\phi} = cot\ a \approx \frac{\overline{P_{\Delta\phi} P_{\Delta}}}{\overline{P_3 P_{\Delta\phi}}} = \frac{\Delta\theta}{\Delta\phi\ cos(\theta)}$$

Because Δu corresponds to and $\Delta\phi$, and because preservation of $\angle\alpha$ entails preservation of its trigonometric ratios:

$$cot\ a \approx \frac{\Delta v}{\Delta u} = \frac{\Delta v}{\Delta\phi}$$

The reason why we focus on the vertical v and replace the horizontal u is because we already know the width of T is fixed at 2π. A constant width means that, given Δu, we already know what it will distort to. It is Δv that we need to find the distortion for. It also means, for this projection, $\Delta u = \Delta\phi$.

This allows us to equate:

$$\frac{\Delta v}{\Delta\phi} = \frac{\Delta\theta}{\Delta\phi\ cos(\theta)}$$

Thus,

$$\frac{\Delta v}{\Delta\phi} = \frac{\Delta\theta}{\Delta\phi\ cos(\theta)}$$

$$\Delta v = \frac{\Delta\theta}{cos(\theta)}$$

$$\frac{\Delta v}{\Delta\theta} = sec\ \theta$$

Now if we let $\Delta\theta$ approach 0, we attain the derivative of v:

$$\frac{dv}{d\theta} = sec\ \theta$$

We then integrate:

$$v(\theta) = \int sec\ \theta\ d\theta = \int \frac{1}{cos\ \theta} d\theta = \int \frac{cos\ \theta}{cos^2\ \theta} d\theta = \int \frac{cos\ \theta}{1 - sin^2\ \theta} d\theta$$

We use u-substitution:

$$u = sin\ \theta$$

$$du = cos\ \theta\ d\theta$$

$$\int sec\ \theta\ d\theta = \int \frac{1}{1 - u^2} du$$

We apply partial fractions:

$$\int \frac{1}{1-u^2} du = \int (\frac{A}{1+u} + \frac{B}{1-u}) du$$

$$A(1 - u) + B(1 + u) = 1$$

When $u = 1, 2B = 1, B = \frac{1}{2}$

When $u = -1, 2A = 1, A = \frac{1}{2}$

$$\int (\frac{\frac{1}{2}}{1+u} + \int \frac{\frac{1}{2}}{1-u}) du$$

$$\frac{1}{2}(log|1 + u| - log|1 - u) + C$$

Then we re-substitute:

$$v(\theta) = \frac{1}{2}(log|1 + sin\,\theta| - log|1 - sin\,\theta|) + C$$

Finally, we simplify:

$$\frac{1}{2}(log|\frac{1+sin\,\theta}{1-sin\,\theta}|) + C$$

$$\frac{1}{2}(log|\frac{1+sin\,\theta}{1-sin\,\theta}|) + C$$

$$\frac{1}{2}(log|\frac{(1+sin\,\theta)^2}{cos^2\theta}|) + C$$

$$log|\frac{1+sin\,\theta}{cos\,\theta}| + C$$

$$v(\theta) = log|sec\,\theta + tan\,\theta| + C$$

Since $v(0) = 0$ (when $\theta = 0$, the projected point $v = 0$), $C = 0$:

$$u(\phi, \theta) = \phi$$

$$v(\phi, \theta) = log|sec\,\theta + tan\,\theta|$$

This can be demonstrated with spherical coordinate $(\phi, \theta) = (-\frac{\pi}{3}, \frac{\pi}{4})$.

$$u = -\frac{\pi}{3} \approx -1.047$$

$$v = log|sec\,(\frac{\pi}{4}) + tan\,(\frac{\pi}{4})| \approx 0.383$$

Spherical coordinates $(\phi, \theta) = (-\frac{\pi}{3}, \frac{\pi}{4})$ project to Cartesian coordinates $(-1.047, 0.383)$.

The most prominent application of the cylindrical projection is in cartography, with the infamous Mercator projection of the Earth. Because angle is preserved, lines with a constant compass bearing (in our example $\angle\alpha$ acted as the bearing and $\overline{P_3P_\Delta}$ as the line) map to straight lines on a two-dimensional plane. This is helpful to navigators, who can simply draw a straight line between two points, measure its compass bearing, and from there rely on only a compass to navigate.

6. Equal-Area Projection

An equal-area projection aims to retain the relative area of the surface of the sphere. As we examined the cylindrical conformal projection in the previous section, we will examine the cylindrical equal-area projection as our example. Similar to its conformal counterpart, this projection maps the sphere onto the inside of a cylinder before rolling it out (Diagram 15). However, instead of aiming to preserve the angle between curves, it aims to preserve the area of a given space. As a result, we will also utilize our same setup of a small rectangle (small angular change Δϕ and Δθ from P_3 and small change in length Δu and Δv on T) in this example, so that we can consider its area.

Because of this same setup, much of the early process of the mathematics of this projection matches with the conformal counterpart. Carrying over the terminology and variables involved in the previous section, we continue to claim that (corresponding to Diagram 17):

$$\overline{P_3P_{\Delta\phi}} = (\overline{O_\theta P_3})(\Delta\phi) = \Delta\phi\, cos(\theta)$$
$$\overline{P_3P_{\Delta\theta}} = (\overline{OP_3})(\Delta\theta) = (1)(\Delta\theta) = \Delta\theta$$

However, in this case instead of finding an equation focused on angle measure in terms of ϕ and θ, we aim to find an equation focused on area. To do this, we set the area of the rectangular space on the sphere equal to the area of the rectangular space on the flat plane. Referring to Diagram 10:

$$\Delta u \times \Delta v = \Delta\phi\, cos\,\theta \times \Delta\theta$$

As with the cylindrical conformal projection, we focus on finding Δv (corresponding to length) because the width (corresponding to Δu) is fixed at 2π—for any given Δu, we already know that the parallel of latitude as a whole has to distort to end up fitting 2π, and it is Δv that we need to determine. As a result, we can replace Δu with Δϕ, and simplify accordingly:

$$\Delta\phi \times \Delta v = \Delta\phi\, cos\,\theta \times \Delta\theta$$
$$\Delta v = cos\,\theta \times \Delta\theta$$
$$\frac{\Delta v}{\Delta\theta} = cos\,\theta$$

Letting Δθ approach 0, we attain the derivative of v:

$$\frac{dv}{d\theta} = cos\,\theta$$

Integrating, we get:

$$v(\theta) = \sin\theta + C$$

Since we know that $v(0) = 0$ (a latitude of 0 projects to the u-axis on the two-dimensional plane), $C = 0$, and we attain our final projection equations:

$$u = \phi$$

$$v = \sin\theta$$

For a potential spherical coordinate $(\phi, \theta) = (\frac{\pi}{3}, \frac{\pi}{4})$:

$$u = \frac{\pi}{3} \approx 1.047$$

$$v = \sin(\frac{\pi}{4}) \approx 0.707$$

Spherical coordinates $(\phi, \theta) = (\frac{\pi}{3}, \frac{\pi}{4})$ project to Cartesian coordinates $(1.047, 0.707)$.

The major advantage of an equal-area projection is simply its defining property—it preserves area. For a projection of the Earth's surface, an equal-area projection is not ideal for navigation because it doesn't preserve angle like the conformal one. However, it does accurately display the relative sizes of different portions of the map. For example, while the Mercator (cylindrical conformal) projection highlights Greenland as a comparable size to the entire African continent, the Lambert cylindrical equal-area projection highlights the true relative sizes, and Greenland is much smaller than Africa.

To connect this to the mathematics we have done, we can see that as compared to the latitude of $\frac{\pi}{4}$ for the cylindrical conformal projection that maps to $v = 0.383$, the equal-area projection maps the same latitude higher to $v = 0.707$. This highlights the distortion of area in the conformal projection resulting from an effort to preserve angle, while for the equal-area projection, v is higher on the map (ex. Greenland is compressed and maps to the very top of the map). Of course, this is reversed for a point in the southern hemisphere, with the equal-area projection at a lower latitude than the conformal.

An application of the map is a *geocode system*. Geocoding a location is the process of assigning a description to that particular place; this can be by address, coordinates, or even a description of the features of that location. The creation of one type of geocode system (creating a description for every point on the earth) involves a grid splitting of the Earth (or a region of the Earth) into portions of equal area, also known as *DGGS cells*. For this, an equal-area projection is needed. A geocode system can be useful for identifying locations by feature and data analysis, for example businesses hoping to optimize their product distribution systems, or keeping record of the population of areas.

7. Conclusion

7.1. Summary of Real World Value

As showcased throughout this paper, there are a multitude of real world applications of three-dimensional to two-dimensional projections, each of which depends on the type of projection. Besides maps of the Earth, where conformal projections are used to optimize navigation and equal-area projections are used for analysis of countless data parameters, projection can also be used to map other planetary bodies, such as the moon. Understanding the use of geometry and calculus in these projections highlights the not only extensive, but indispensably crucial application of mathematics in the real world.

7.2. Limitations

It is important to note that many of these applications involve, in reality, much more complex considerations. As a major example, any projection discussed in this paper assumes that the Earth is an exact sphere. However, this is not true; the shape of Earth is in fact irregular, meaning that the projections must be tweaked in order to accurately represent the Earth's surface. Additionally, technology, such as the Global Positioning System (GPS) and advanced satellite imagery, has greatly advanced our understanding of the Earth's surface, making several of these projections obsolete, as more accurate maps are able to be pieced together. Nonetheless, the importance of the mathematics of projection is impossible to deny, even in the modern era.

Bibliography

Briney, Amanda. "A Look at the Mercator Projection." *GIS Lounge*, 13 Jan. 2014, https://www.gislounge.com/look-mercator-projection/#:~:text=The%20Mercator%20projection%20was%20originally,Ptolemy%27s%20latitude%20and%20longitude%20grid.

Daners, Daniel. "The Mercator and Stereographic Projections, and Many in Between." *The American Mathematical Monthly*, vol. 119, no. 3, 29 July 2011, pp. 1–14., https://doi.org/10.4169/amer.math.monthly.119.03.199.

Esri. "ArcMap." *What Is Geocoding?*, 2021, https://desktop.arcgis.com/en/arcmap/latest/manage-data/geocoding/what-is-geocoding.htm#:~:text=Geocoding%20is%20the%20process%20of,at%20once%20in%20a%20table.

Harley, J. B. "The Map and the Development of the History of Cartography." *University of Chicago*, 1987, https://press.uchicago.edu/books/HOC/HOC_V1/HOC_VOLUME1_chapter1.pdf.

Israel, Robert. "Mercator's Projection." *Mercator's Projection*, The University of British Columbia, 20 Jan. 2003, https://personal.math.ubc.ca/~israel/m103/mercator/mercator.html.

Manson, Steven, et al. "Scale and Projections." *Mapping, Society, and Technology*, Open Textbook Library, Minneapolis, MN, 2017.

National Oceanic and Atmospheric Administration. "Is the Earth Round?" *NOAA's National Ocean Service*, 26

Feb. 2021, https://oceanservice.noaa.gov/facts/earth-round.html.

Osserman, Robert. "Mathematical Mapping from Mercator to the Millenium." *Mathematical Sciences Research Institute*, 25 Mar. 2004, pp. 233–258., https://doi.org/10.1090/spec/040/23.

Polking, John C. "Mapping the Sphere." *Rice University*, Department of Mathematics, 28 Jan. 2000, https://math.rice.edu/~polking/cartography/cart.pdf.